Jordan curve theorem

Let Q be the unit square. Consider a sequence of continuous paths $f_n : [0,1] \to \mathbb{R}^2$ with the following properties:

(1) for any $t \in [0,1]$, we have $\rho(f_n(t), f_{n+1}(t)) \leqslant 2^{-n}$;

(2) for any $x \in Q$ and any $n = 1, 2, \ldots$, there is some $s \in [0,1]$ with $\rho(x, f_n(s)) \leqslant 2^{-n+1}$.

An example of such sequence was given in class.

Problem 1. For every $t \in [0,1]$, set $f(t) = \lim_{n\to\infty} f_n(t)$. Prove that this limit exists. Prove that the map f thus defined is (uniformly) continuous.

Problem 2. Prove that the f-image of $[0,1]$ coincides with Q.

A subset $U \subset \mathbb{R}^2$ is said to be *open* if, for every $x \in U$, there exists $\varepsilon > 0$ such that $y \in U$ whenever $\rho(x,y) < \varepsilon$. An open subset U is said to be (path) connected if, for all x, $y \in U$, there is a continuous path $f : [0,1] \to U$ with $f(0) = x$ and $f(1) = y$.

Problem 3. Prove that U is connected if and only if U cannot be represented as a disjoint union of two open subsets.

Problem 4. Suppose that U is connected. Prove that any two points in U can be connected by a broken line lying entirely in U.

Let P be any concatenation of oriented line segments $I_0, \ldots, I_{n-1}$ such that I_{k+1} shares an endpoint with I_k. We set $I_n = I_0$. We do not assume that P is not self-intersecting. One can still define the *winding number* $W(a) = W_P(a)$ using some direction different from the directions of all edges of P.

Problem 5. Prove that $W(a)$ does not depend on the choice of a direction.

Problem 6. Prove that the function W is constant on every broken line disjoint from P.