Groups; Permutation groups

Lecture 2

Permutation groups.

PROBLEM 1. Prove that for $n > 2$ the permutation group $\mathbf{S}_n$ is not Abelian.

PROBLEM 2. For what kind of permutations from $\mathbf{S}_n$ do all permutations commute with them?

PROBLEM 3. Find products of permutations:

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 1 & 5 & 2 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 7 & 6 & 5 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 2 & 1 & 8 & 7 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$;

c) $\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ n & 1 & 2 & \dots & n-3 & n-2 & n-1 \end{pmatrix}$.

PROBLEM 4. Find the orders of permutations from $\mathbf{S}_n$:

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 1 & 5 & 2 & 6 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 2 & 1 & 8 & 7 & 9 \end{pmatrix}$;

c) $\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & 1 \end{pmatrix}$.

DEFINITION 1. An *inversion* for a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ i_1 & i_2 & i_3 & \dots & i_{n-1} & i_n \end{pmatrix}$$

is a pair $k, l \in \{1, \dots, n\}$ with the properties $k < l$ and $i_k > i_l$.

PROBLEM 5. Find all inversions in the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 2 & 1 & 8 & 7 & 9 \end{pmatrix}.$$

DEFINITION 2. *Parity* of a permutation $\sigma \in \mathbf{S}_n$ if the parity of the number of inversions in it.

PROBLEM 6. Define the parity of the following permutations:

a) $\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ n & n-1 & n-2 & \dots & 3 & 2 & 1 \end{pmatrix}$;

c) $\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ n & 2 & 3 & \dots & n-2 & n-1 & 1 \end{pmatrix}$.

Definition 3. Permutation swapping only two elements and leaving all other elements, is called a *transposition*. If a transposition swaps elements i and j, it is denoted just by (i, j).

Problem 7. Multiplying a permutation (from the right or left side) to an arbitrary transposition changes its parity to the opposite one.

Problem 8. Are there even or odd permutations more in the group $\mathbf{S}_n$?

Problem 9. Prove that every permutation can be decomposed as the product of transpositions.

Problem 10. Prove that every permutation can be decomposed as the product of permutations of only the following form:

a) transpositions $(1,2)$, $(2,3)$, $\dots$, $(i,i+1)$, $\dots$, $(n-1,n)$;

b) transpositions $(1,2)$, $(1,3)$, $\dots$, $(1,i)$, $\dots$, $(1,n)$;

c) only two permutations: the transposition $(1,2)$ and the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & 1 \end{pmatrix}.$$

Problem 11. Prove that under multiplication of permutations their parities are added modulo two.

Problem 12. What of following subsets of the group $\mathbf{S}_n$ are subgroups in it:

a) the set of all even permutations (notation: $\mathbf{A}_n$);

b) the set of all odd permutations;

c) the set of all permutations, leaving elements of some subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$;

d) the set of all permutations where images of some subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ belong to this subset;

e) the set $\{E, (12)(34), (13)(24), (14)(23)\}$ (notation: $\mathbf{V}_4$);

f) the set $\{E, (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$ (notation: $\mathbf{D}_4$)?

Definition 4. A *cycle* $(i_1\, i_2\, \dots i_{k-1}\, i_k)$ is a permuation which maps i_1 into i_2, i_2 into i_3, $\dots$, i_{k-1} into i_k, i_k into i_1, and does not move other elements.

Two cycles $(i_1,\ i_2\ \dots i_{p-1}\, i_p)$ and $(j_1\, j_2\ \dots j_{q-1}\, j_q)$ are called *independent*, if the sets $\{i_1, \dots, i_p\}$ and $\{j_1, \dots, j_q\}$ do not intersect.

Problem 13. Any permutation can be uniquely decomposed into the product of independent cycles.

Problem 14. Find the parity of a permutation decomposed into the product of cycles of lengths $l_1, \dots, l_q$.

Problem 15. Find the order of a permutation decomposed into the product of cycles of lengths $l_1, \dots, l_q$.

Problem 16. Find the product of two permutations decomposed into the product of independent cycles:

a) $(1\,2\,3\,4)(5\,6\,7)(8\,9) \cdot (1\,2)(3\,4\,5)(6\,7\,8\,9)$;

b) $(1\,2)(3\,4)(5\,6)(7\,8) \cdot (1\,8)(2\,7)(3\,6)(4\,5)$.

PROBLEM 17. Prove that two cycles $\{i_1, \dots, i_p\}$ and $\{j_1, \dots, j_q\}$ commute if and only if either these cycles are independent or one of them is the power of another one.

PROBLEM 18. Permutations of even or odd order are there more in the group $\mathbf{S}_n$?