# Groups; permutation groups

## Lecture 3

## Problems with permutations; Isometry groups of polyhedrons.

Problem 1 (Kelley Theorem). Prove thath every finite group can be embedded into a permutation group $\mathbf{S}_n$ for some $n$, not greater than its order.

Problem 2. What is maximal possible order of a permutation from $\mathbf{S}_{13}$?

Problem 3. What is maximal number of mutually commuting permutations in the group $\mathbf{S}_6$?

Problem 4. What permutations can be represented as

a) a product of some cycles of the length 3;
b) a product of cycles $(1\,2\,3),\ (1\,2\,4),\ (1\,2\,5), \dots,\ (1\,2\,n)$;
c) a product of two cycles?

Problem 5. Suppose that in the decomposition of $\sigma \in \mathbf{S}_n$ into disjoint cycles the number of cycles of the length $i$ is $m_i$. For what $m_2, \dots, m_n$ can we find such a substitution $\tau$ that $\tau^2 = \sigma$?

Problem 6. If in the game "tap" one swaps gaming pieces with the numbers 14 and 15, then following game rules it is impossible to get the initial disposition of pieces.

Definition 1. For every permutation $\sigma \in \mathbf{S}_n$ and every $i \in \{1, \dots, n\}$ by $m_i$ we denote the number of elements in $\sigma$, greater than $i$ and replaced from the left side of $i$. The table $(m_1, m_2, \dots, m_n)$ is called a *table of inversions* of $\sigma$.

Problem 7. What permutation has the table of inversions equal to $(0, 0, ..., 0)$? Equal to $(n-1, n-2, ..., 0)$?

Problem 8. What tables of inversions can have permutations of the length $n$? How many different tables are there?

Problem 9. Prove that between all permutations from $\mathbf{S}_n$ and their tables of inversions there exists a one-to-one correspondence. Find an algorithm reconstructing a permutation by its table of inversions. What is the complexity of this algorithm?

Problem 10. Find an average number of inversions in a permutation of the length $n$.

Definition 2. For an arbitrary permutation $\sigma \in \mathbf{S}_n$ we call its element $i$ a *maximal element from left to right*, if there are no greater elements from its left side (for example for the permutation $(1\,2\,3\,4\,5)$ every element is of this type, and in $(1\,3\,2\,5\,4)$ the elements 1, 3 and 5 are of this type, other elements are not).

Problem 11. What is the average number of maximal from left to right elements in a random permutation of the length $n$?

Problem 12. For any figure (any set of points) in a finitely dimensional affine space $\mathbb{R}^n$ the set of all its isometries (transformations preserving distances between points) forms a group under the composition law.

Problem 13. Prove that the isometry group of a square is isomorphic to $\mathbf{D}_4$.

Definition 3. Isometry group of a regular $n$-polygon is denoted by $\mathbf{D}_n$ and is called a *dihedral group*.

Problem 14. How many elements are there in the group $\mathbf{D}_n$? What is the minimal number of generators of this group?

Problem 15. Draw figures on the plane with the isometry group isomorphic to
a) $\mathbb{Z}_2$; b) $\mathbf{V}_4$; c) $\mathbb{Z}_3$; d) $\mathbb{Z}_n$.

Problem 16. Find groups of proper isometries of
a) cube; b) octahedron ; c)$^*$ icosahedron.